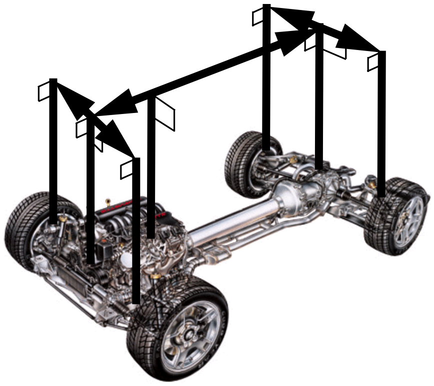
Torsion of Shafts

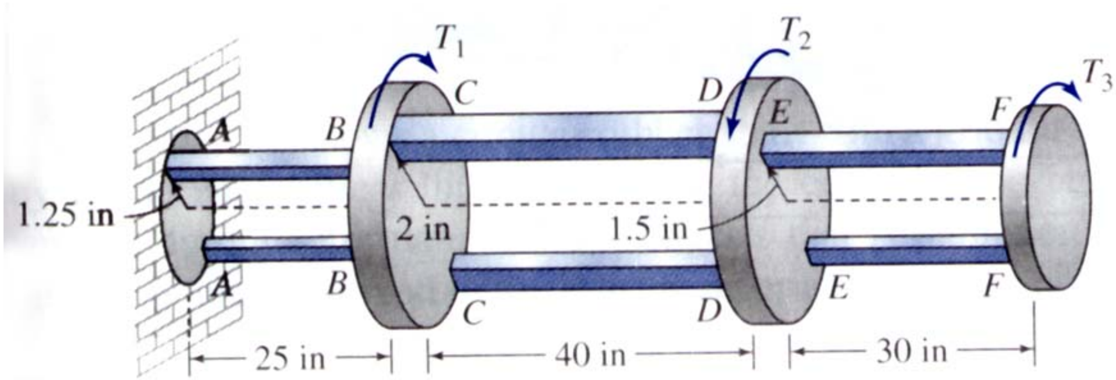
- Shafts are structural members with length significantly greater than the largest cross-sectional dimension used in transmitting torque from one plane to another.



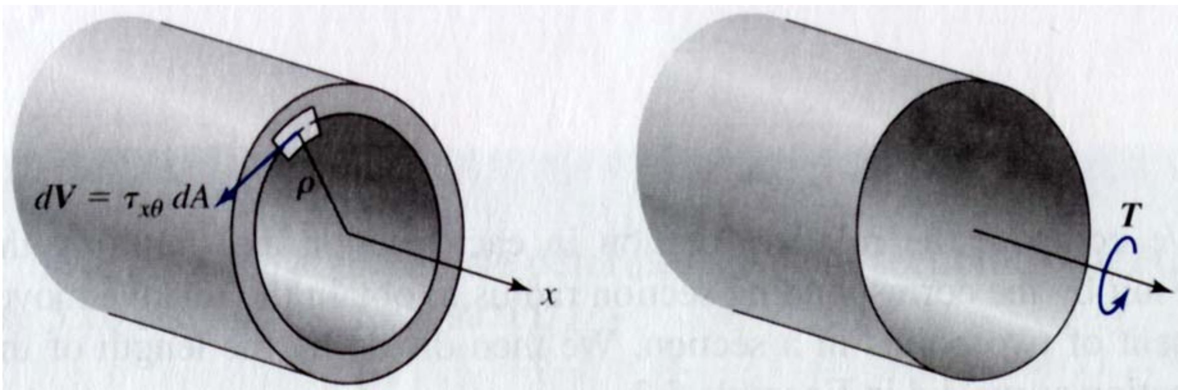
Learning objectives

- Understand the theory, its limitations and its applications for design and analysis of Torsion of circular shafts.
- Develop the discipline to visualize direction of torsional shear stress and the surface on which it acts.

5.4 Three pairs of bars are symmetrically attached to rigid discs at the radii shown. The discs were observed to rotate by angles $\phi_1 = 1.5°$, $\phi_2 = 3.0°$, and $\phi_3 = 2.5°$ in the direction of the applied torques T_1, T_2, and T_3 respectively. The shear modulus of the bars is 40 ksi and the area of cross-section is 0.04 in^2. Determine the applied torques.



Internal Torque



$$T = \int_A \rho \, dV = \int_A \rho \tau_{x\theta} dA \qquad 5.1$$

- Equation is independent of material model as it represents static equivalency between shear stress and internal torque on a cross-section

5.11 A hollow titanium (G_{Ti} = 36 GPa) shaft and a hollow Aluminum (G_{Al} = 26 GPa) shaft are securely fastened to form a composite shaft as shown in Fig. P5.11. The shear strain $\gamma_{x\theta}$ in polar coordinates at the section is $\gamma_{x\theta} = 0.05\rho$ where ρ is in meters and the dimensions of the cross-section are d_i= 40 mm, d_{Al}= 80 mm and d_{Ti} = 120 mm. Determine the equivalent internal torque acting at the cross-section.

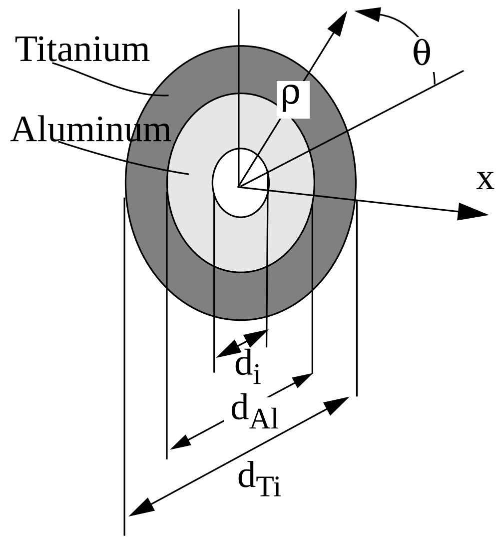
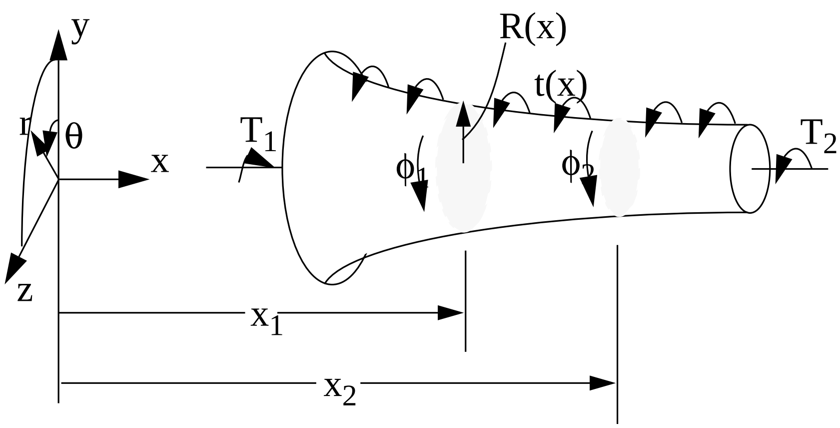


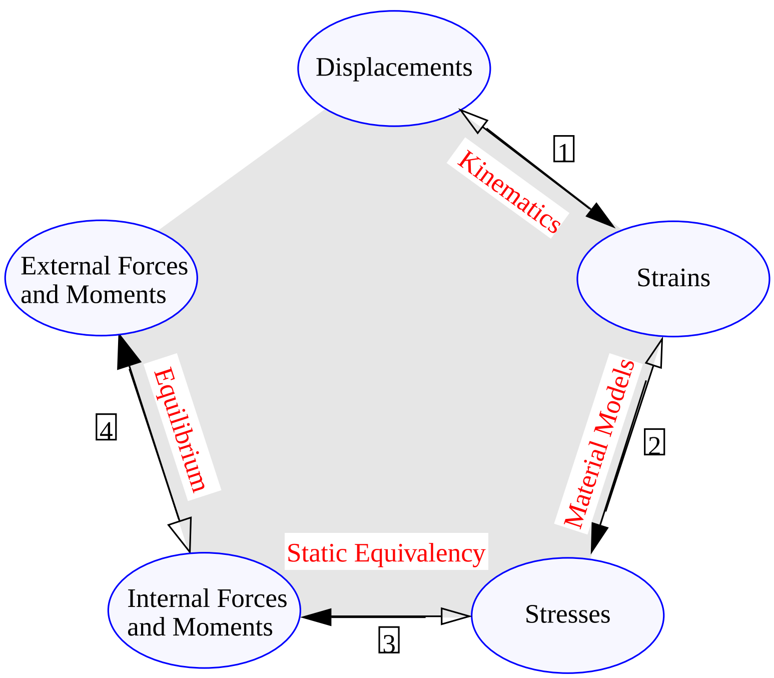
Fig. P5.11

Theory for Circular Shafts

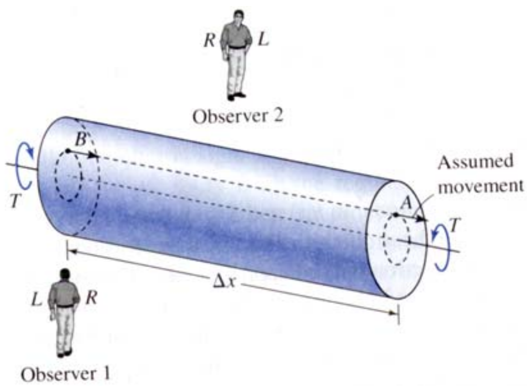
Theory Objective

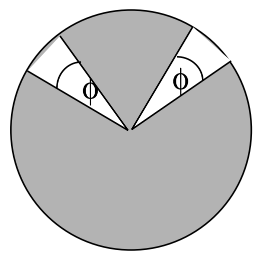
- (i) to obtain a formula for the relative rotation $(\phi_2 - \phi_1)$ in terms of the internal torque ***T***.
- (ii) to obtain a formula for the shear stress $\tau_{x\theta}$ in terms of the internal torque ***T***.





Kinematics

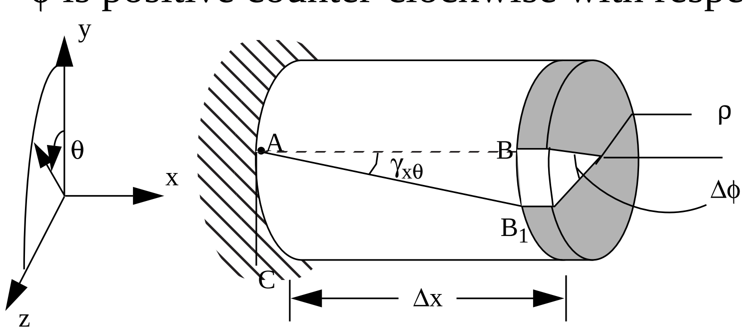


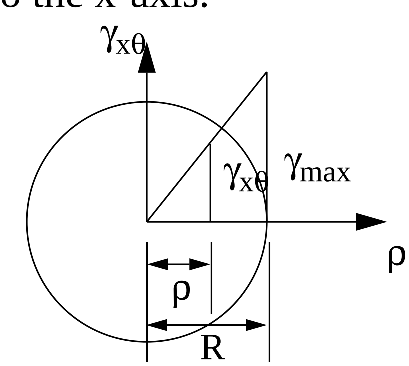


Assumption 1	Plane sections perpendicular to the axis remain plane during deformation. (No Warping)
Assumption 2	On a cross-section, all radials lines rotate by equal angle during deformation.
Assumption 3	Radials lines remain straight during deformation.

$$\phi = \phi(x)$$

- ϕ is positive counter-clockwise with respect to the x-axis.





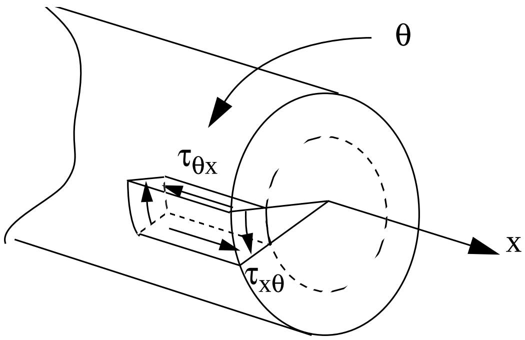
Assumption 4 Strains are small. $\gamma_{x\theta} = \rho\dfrac{d\phi}{dx}$

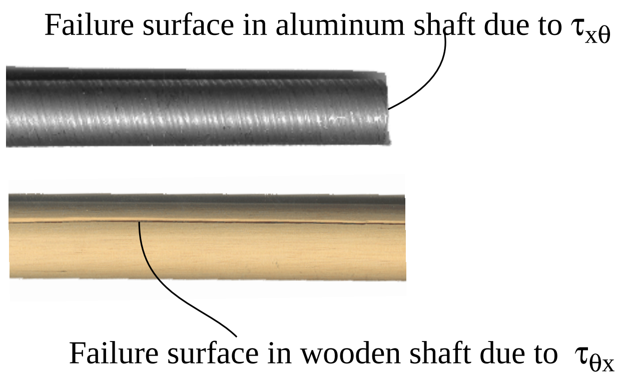
Material Model

Assumption 5 Material is linearly elastic.

Assumption 6 Material is isotropic.

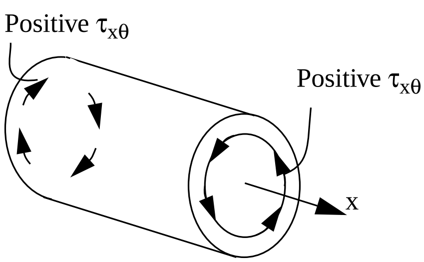
From Hooke's law $\tau = G\gamma$, we obtain: $\tau_{x\theta} = G\rho\dfrac{d\phi}{dx}$

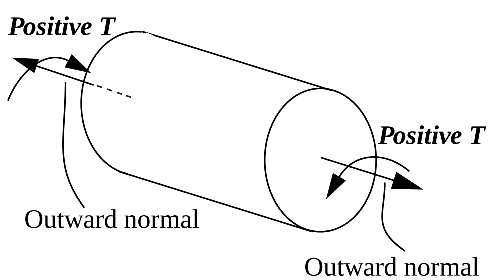




Sign Convention

- Internal torque is considered positive if it is counter-clockwise with respect to the outward normal to the imaginary cut surface.





Torsion Formulas

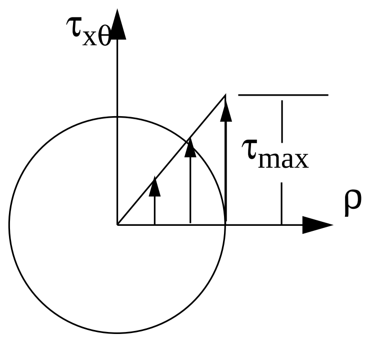
$$T = \int_A G\rho^2 \frac{d\phi}{dx} dA = \frac{d\phi}{dx}\int_A G\rho^2 dA$$

Assumption 7 Material is homogenous across the cross-section.

$$\frac{d\phi}{dx} = \frac{T}{GJ}$$

- J is the polar moment of inertia for the cross-section.
- The quantity GJ is called the torsional rigidity.
- Circular cross-section of radius R or diameter D, $J = \frac{\pi}{2}R^4 = \frac{\pi}{32}D^4$.

$$\tau_{x\theta} = \frac{T\rho}{J}$$



$$\phi_2 - \phi_1 = \int_{\phi_1}^{\phi_2} d\phi = \int_{x_1}^{x_2} \frac{T}{GJ} dx$$

Assumption 8 Material is homogenous between x_1 and x_2.

Assumption 9 The shaft is not tapered.

Assumption 10 The external (hence internal) torque does not change with x between x_1 and x_2.

$$\phi_2 - \phi_1 = \frac{T(x_2 - x_1)}{GJ}$$

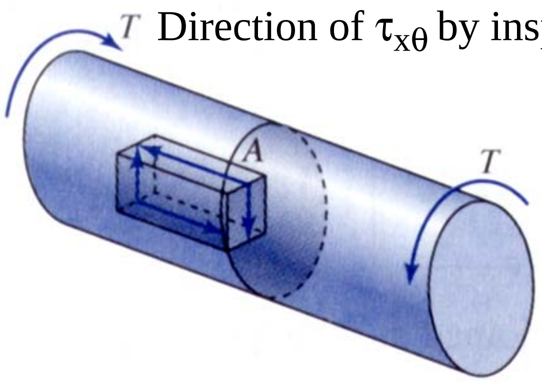
Two options for determining internal torque ***T***

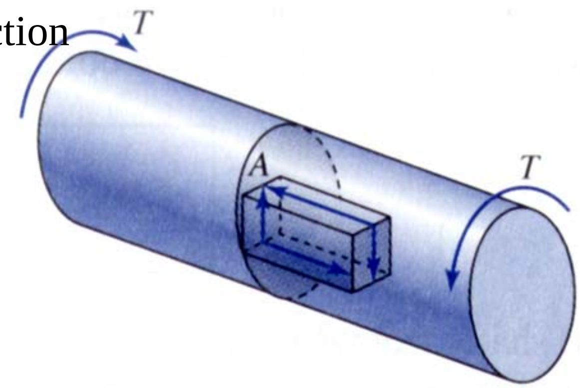
- ***T*** is always drawn in counter-clockwise direction with respect to the outward normal of the imaginary cut on the free body diagram.
  - Direction of $\tau_{x\theta}$ can be determined using subscripts.
  - Positive ϕ is counter-clockwise with respect to x-axis.
  - $\phi_2 - \phi_1$ is positive counter-clockwise with respect to x-axis
- ***T*** is drawn at the imaginary cut on the free body diagram in a direction to equilibrate the external torques.
  - Direction of $\tau_{x\theta}$ must be determined by inspection.
  - Direction of ϕ must be determined by inspection.
  - Direction of $\phi_2 - \phi_1$ must be determined by inspection.

Torsional Stresses and Strains

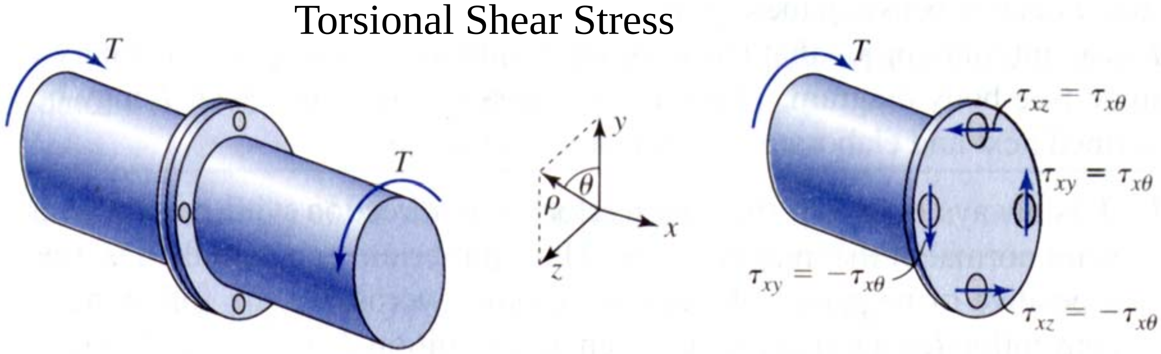
- In polar coordinates, all stress components except $\tau_{x\theta}$ are assumed zero. Shear strain can be found from Hooke's law.

Direction of $\tau_{x\theta}$ by inspection

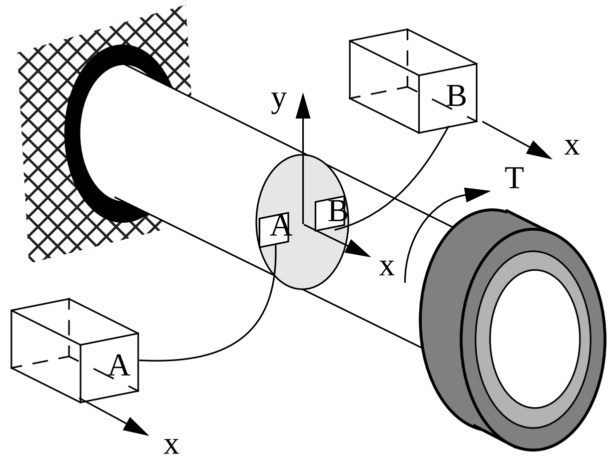




Torsional Shear Stress

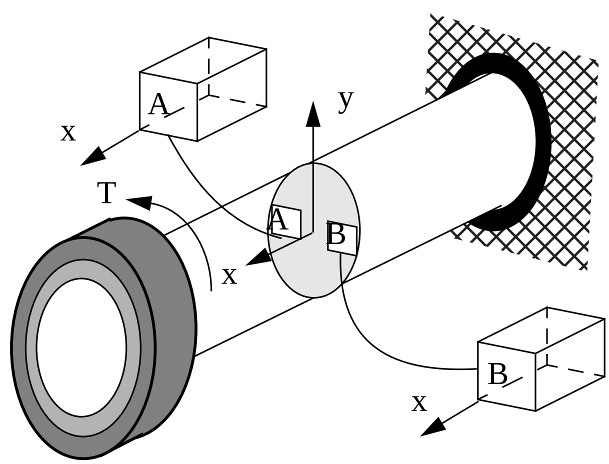


5.15 Determine the direction of shear stress at points A and B (a) by inspection, and (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy}.

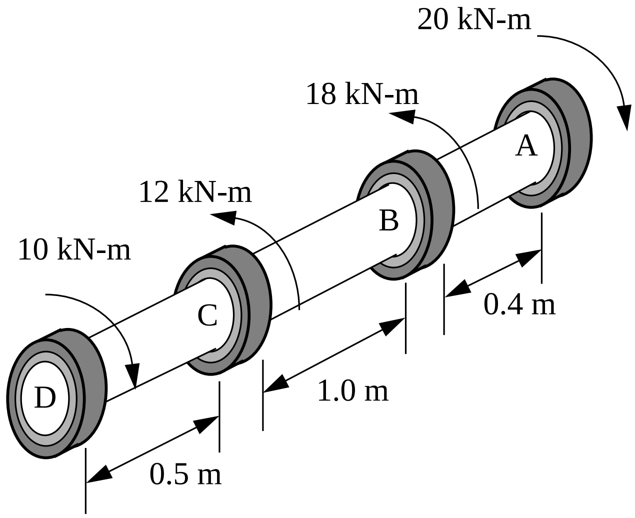


Class Problem 1

5.16 Determine the direction of shear stress at points A and B (a) by inspection, and (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy}.



5.21 Determine the internal torque in the shaft below by making imaginary cuts and drawing free body diagrams.

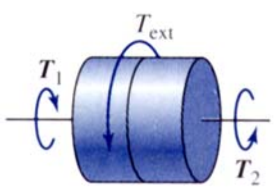


Torque Diagram

- A torque force diagram is a plot of internal torque ***T*** vs. x
- Internal torque jumps by the value of the external torque as one crosses the external torque from left to right.
- An torsion template is used to determine the direction of the jump in ***T.***

A template is a free body diagram of a small segment of a shaft created by making an imaginary cut just before and just after the section where the external torque is applied.

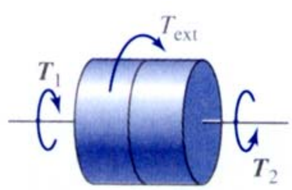
Template 1



Template 1 Equation

$$T_2 = T_1 - T_{ext}$$

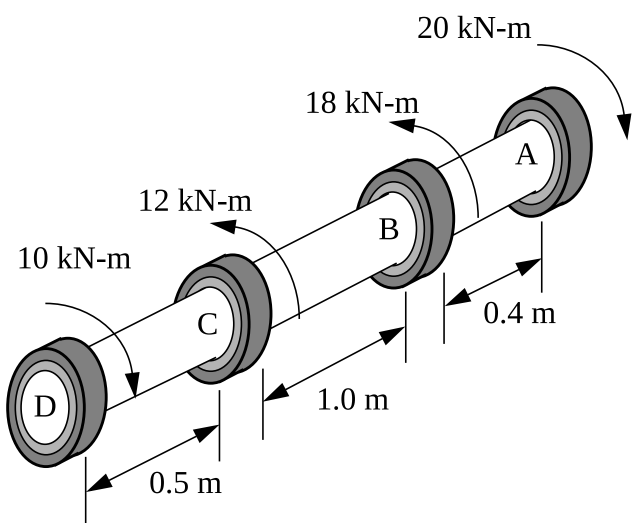
Template 2



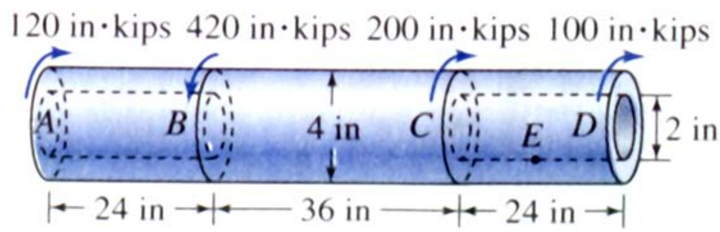
Template 2 Equation

$$T_2 = T_1 + T_{ext}$$

5.21 Determine the internal torque in the shaft below by drawing the torque diagram.



5.28 A solid circular steel (G_S = 12,000 ksi) shaft BC is securely attached to two hollow steel shafts AB and CD as shown. Determine: (a) the angle of rotation of section at D with respect to section at A. (b) the maximum torsional shear stress in the shaft (c) the torsional shear stress at point E and show it on a stress cube. Point E is on the inside bottom surface of CD.



5.32 The external torque on a drill bit varies as a quadratic function to a maximum intensity of q in.lb/in as shown. If the drill bit diameter is d, its length L, and modulus of rigidity G, determine (a) the maximum shear stress on the drill bit. (b) the relative rotation of the end of the drill bit with respect to the chuck.

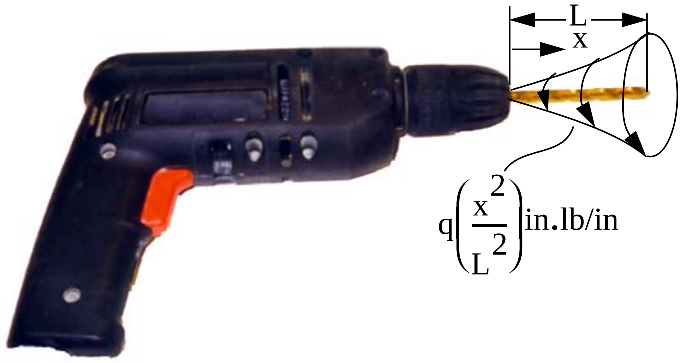


Fig. P5.32

Statically Indeterminate Shafts

- Both ends of the shaft are built in, leading to two reaction torques but we have only one moment equilibrium equation.
- The compatibility equation is that the relative rotation of the right wall with respect to the left wall is zero.
- Calculate relative rotation of each shaft segment in terms of the reaction torque of the left (or right) wall. Add all the relative rotations and equate to zero to obtain reaction torque.

5.55 Two hollow aluminum (G = 10,000 ksi) shafts are securely fastened to a solid aluminum shaft and loaded as shown Fig. P5.55. Point E is on the inner surface of the shaft. If T= 300 in-kips in Fig. P5.55, Determine (a) the rotation of section at C with respect to rotation the wall at A. (b) the shear strain at point E.

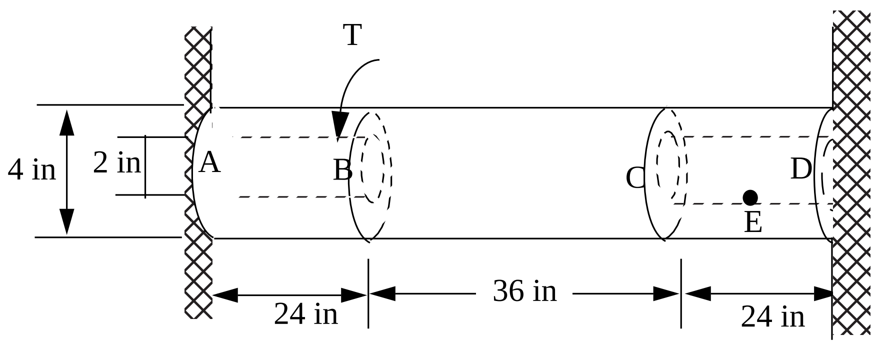


Fig. P5.55

5.61 Under the action of the applied couple the section B of the two tubes shown Fig. P5.61 rotate by an angle of 0.03 rads. Determine (a) the magnitude maximum torsional shear stress in aluminum and copper. (b) the magnitude of the couple that produced the given rotation.

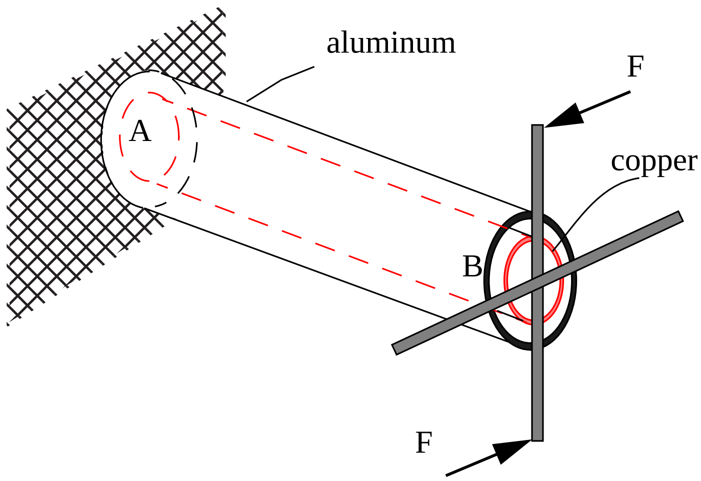


Fig. P5.61

Composite Shafts

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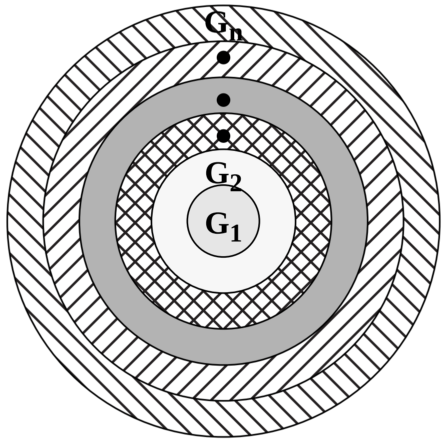


- Assumption 7 of material homogeneity across the cross-section is no longer valid.

$$T = \frac{d\phi}{dx}\int_A G\rho^2 dA = \frac{d\phi}{dx}\left[\int_{A_1} G_1\rho^2 dA + \int_{A_2} G_2\rho^2 dA + \cdot + \cdot + \int_{A_n} G_n\rho^2 dA\right]$$

- Rest of derivation is same as for homogenous cross-section.

Homogenous cross-section	Composite cross-section
$\frac{d\phi}{dx} = T/[GJ]$	$\frac{d\phi}{dx} = T/\left[\sum_{j=1}^{n} G_j J_j\right]$
$\phi_2 - \phi_1 = T(x_2 - x_1)/[GJ]$	$\phi_2 - \phi_1 = T(x_2 - x_1)/\left[\sum_{j=1}^{n} G_j J_j\right]$
$\tau_{x\theta} = (T\rho)/J$	$(\tau_{x\theta})_i = G_i \rho T/\left[\sum_{j=1}^{n} G_j J_j\right]$

5.61 Resolve Problem 5.61 using formulas for composite shaft. Under the action of the applied couple the section B of the two tubes shown Fig. P5.61 rotate by an angle of 0.03 rads. Determine (a) the magnitude maximum torsional shear stress in aluminum and copper. (b) the magnitude of the couple that produced the given rotation.

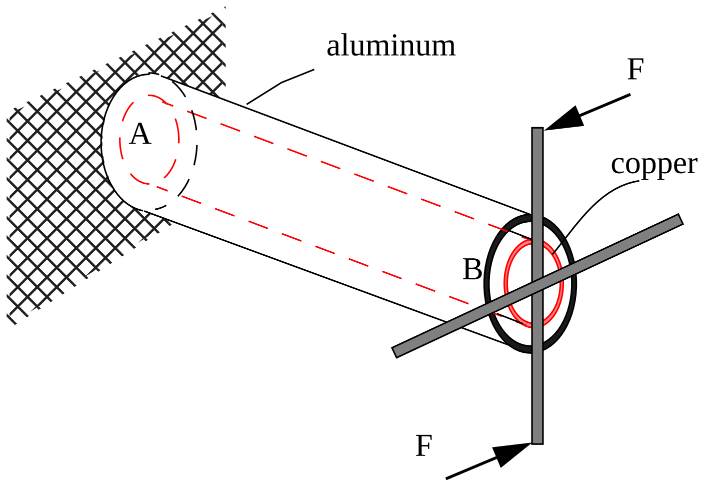
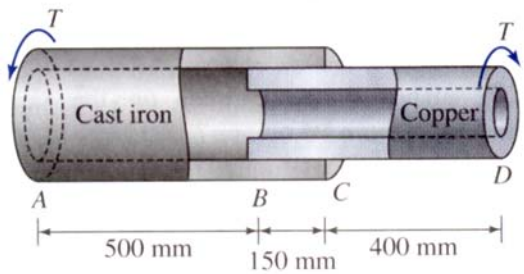


Fig. P5.61

Class Problem 2

A cast iron pipe (G_{ir} = 70 GPa) and a copper pipe (G_{cu} = 40 GPa) are securely bonded together as shown. The outer diameters of the two pipes are 50 mm and 70 mm and wall thickness of each pipe is 10 mm. The applied torque is T= 2000 N.m



- Write the formulas in terms of variables you would use in each segment of the shaft.
- What is the value of internal torque you would use in each segment of the shaft?
- What is the value of ρ you would use in segment BC to find maximum torsional shear stress in each material?

5.77 A 1-m long shaft is to be designed to transmit a torque of 3300 N-m. The outside diameter of the shaft must be 40 mm to fit existing attachments. The shaft can be all aluminum, all titanium, or a composite of the two material. The Shear Modulus of rigidity G, the allowable shear stress τ_{allow}, and the density γ of titanium and aluminum are given in table below. Determine the diameters to the nearest millimeter of the *lightest* shaft and the corresponding mass.

Material	G GPa	τ_{allow} MPa	γ Mg/m^3
Titanium Alloy	36	300	4.4
Aluminum	28	180	2.8