Hilbert's paradox of the Grand Hotel

Hilbert's paradox of the Grand Hotel is a veridical paradox (a valid argument with a seemingly absurd conclusion, as opposed to a falsidical paradox, which is a seemingly valid demonstration of an actual contradiction) about infinite sets presented by David Hilbert in the 1920s, meant to illustrate certain counterintuitive properties of infinite sets.

The paradox

Consider a hypothetical hotel with a countably infinite number of rooms, all of which are occupied. One might be tempted to think that the hotel would not be able to accommodate any newly arriving guests, as would be the case with a finite number of rooms.

Finitely many new guests

Suppose a new guest arrives and wishes to be accommodated in the hotel. Because the hotel has infinitely many rooms, we can move the guest occupying room 1 to room 2, the guest occupying room 2 to room 3 and so on, and fit the newcomer into room 1. By repeating this procedure, it is possible to make room for any finite number of new guests.

Infinitely many new guests

It is also possible to accommodate a countably infinite number of new guests: just move the person occupying room 1 to room 2, the guest occupying room 2 to room 4, and, in general, the guest occupying room \( n \) to room \( 2n \), and all the odd-numbered rooms (which are countably infinite) will be free for the new guests.

Infinitely many coaches with infinitely many guests each

It is possible to accommodate countably infinitely many coachloads of countably infinite passengers each, by several different methods. Most methods depend on the seats in the coaches being already numbered (alternatively, the hotel manager must have the axiom of countable choice at his or her disposal). In general any pairing function can be used to solve this problem. For each of these methods, consider a passenger's seat number on a coach to be \( i \), and their coach number to be \( c \), and the numbers \( i \) and \( c \) are then fed into the two arguments of the pairing function.

Prime powers method

Empty the odd numbered rooms by sending the guest in room \( i \) to room \( 2i \), then put the first coach's load in rooms \( 3^n \), the second coach's load in rooms \( 5^n \); for coach number \( c \) we use the rooms \( p^n \)where \( p \) is the \( c \)th odd prime number. This solution leaves certain rooms empty (which may or may not be useful to the hotel); specifically, all odd numbers that are not prime powers, such as 15 or 847, will no longer be occupied. (So, strictly speaking, this shows that the number of arrivals is less than or equal to the number of vacancies created. It is easier to show, by an independent means, that the number of arrivals is also greater than or equal to the number of vacancies, and thus that they are equal, than to modify the algorithm to an exact fit.) (The algorithm works equally well if one interchanges \( n \) and \( c \), but whichever choice is made, it must be applied uniformly throughout.)
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**Interleaving method**

For each passenger, compare the lengths of $n$ and $c$ as written in decimal. (Treat each hotel resident as being in Bus #0.) If either number is shorter, add leading zeroes to it until both values have the same number of digits. Interleave the digits to produce a room number: its digits will be [first digit of bus number]-[first digit of seat number]-[second digit of bus number]-[second digit of seat number]-etc. The hotel (coach #0) guest in room number 1729 moves to room 01070209 (i.e., room 1,070,209.) The passenger on seat 1234 of coach 789 goes to room 01728394 (or just 1728394).

Unlike the prime powers solution, this one fills the hotel completely, and we can extrapolate a guest's original bus and seat by reversing the interleaving process. First add a leading zero if the room has an odd number of digits. Then de-interleave the number into two numbers: the seat number consists of the odd-numbered digits and the bus number is the even-numbered ones. Of course, the original encoding is arbitrary, and the roles of the two numbers can be reversed (seat-odd and bus-even), so long as it is applied consistently.

**Triangular number method**

Those already in the hotel will be moved to room \( \frac{n^2 + n}{2} \), or the \( n \)th triangular number. Those in a coach will be in room \( \frac{(c + n)^2 + c - n}{2} \), or the \( (c + n - 1) \)th triangular number, plus \( (c + n) \). In this way all the rooms will be filled by one, and only one, guest.

This paring function can be demonstrated visually by structuring the hotel as a one-room-deep, infinitely tall pyramid. The pyramid's topmost row is a single room: room 1; its second row is rooms 2 and 3; and so on. The column formed by the set of rightmost rooms will correspond to the triangular numbers. Once they are filled (by the hotel's redistributed occupants), the remaining empty rooms form the shape of a pyramid exactly identical to the original shape. Thus, the process can be repeated for each infinite set.

**Further layers of infinity**

Suppose the hotel is next to an ocean, and an infinite number of aircraft carriers arrive, each bearing an infinite number of coaches, each with an infinite number of passengers. This is a situation involving three "levels" of infinity, and it can be solved by extensions of any of the previous solutions.

The prime power solution can be applied with further exponentiation of prime numbers, resulting in very large room numbers even given small inputs. For example, the passenger in the second seat of the third bus on the second aircraft carrier (address 2-3-2) would raise the 2nd odd prime (5) to 49, which is the result of the 3rd odd prime (7) being raised to the power of his seat number (2). This seat number would have over thirty decimal digits.

The interleaving method can be used with three interleaved "strands" instead of two. The passenger with the address 2-3-2 would go to room 232, while the one with the address 4935-198-82217 would go to room #008,402,912,391,587 (the leading zeroes can be removed).

Anticipating the possibility of any number of layers of infinite guests, the hotel may wish to assign rooms such that no guest will need to move, no matter how many guests arrive afterward. One solution is to convert each arrival's address into a binary number in which ones are used as separators at the start of each layer, while a number within a given layer (such as a guests' coach number) is represented with that many zeroes. Thus, a guest with the prior address 2-5-1-3-1 (five infinite layers) would go to room 10010000010100010 (decimal 295458).

As an added step in this process, one zero can be removed from each section of the number; in this example, the guest's new room is 101000011001 (decimal 2585). This ensures that every room could be filled by a hypothetical guest. If no infinite sets of guests arrive, then only rooms that are a power of two will be occupied.
Infinite layers of nesting

Although a room can be found for any finite number of nested infinities of people, the same is not always true for an infinite number of layers, even if a finite number of elements exists at each layer. For example, suppose some people arrive in a set of flying saucer spaceships which are nested in accordance to the following rules: the smallest ships, each 100 cubic meters in volume, contain ten people. After this, every ship (of any size) is grouped with nine other ships of the same size, inside a mothership exactly 100 times the volume of each of its ten daughter ships. All ships of the same size are isomorphic to one another; for example, each 1,000,000-cubic-meter ship contains exactly ten 10,000-cubic-meter ships, each of which contains exactly ten 100-cubic-meter ships, each containing ten people. This extends upward infinitely, so that there is no "largest ship".

A given passenger's address in this system would be infinite in length, corresponding to the decimal form of one of the real numbers ranging from 0 (address 0-0-0...) to 1 (address 9-9-9...). Exactly one guest would have the address corresponding to one-half, and another the address corresponding to pi. The set of real numbers, and the set of guests in this example, is uncountably infinite. Because no one-to-one pairing can be made between countable and uncountable sets, room at the hotel cannot be made for all of these guests, although any countably infinite subset of them can still be accommodated -- for example, the set of guests whose addresses terminate in an infinitely repeating sequence, corresponding to a rational number.

If this variant is modified in certain ways, then the set of people is countable again. For example, suppose there were a largest ship, directly containing a finite (or infinite) number of both ships and people, and each of these ships in turn contained both ships and people, and so forth. This time, any given person is a finite number of levels "down" from the top, and thus can be identified with a unique finite address. The set of people is countable again, even if the total number of layers is infinite, because we do not have to consider an "infinitieth layer" in either direction.

Analysis

These cases constitute a paradox not in the sense that they entail a logical contradiction, but in the sense that they demonstrate a counter-intuitive result that is provably true: the statements "there is a guest to every room" and "no more guests can be accommodated" are not equivalent when there are infinitely many rooms (an analogous situation is presented in Cantor's diagonal proof).[^1]

This state of affairs might seem, at first blush, to be counter-intuitive. The properties of "infinite collections of things" are quite different from those of "finite collections of things". The paradox of Hilbert's Grand Hotel can be understood by using Cantor's theory of Transfinite Numbers. Thus, while in an ordinary (finite) hotel with more than one room, the number of odd-numbered rooms is obviously smaller than the total number of rooms. However, in Hilbert's aptly named Grand Hotel, the quantity of odd-numbered rooms is not smaller than total "number" of rooms. In mathematical terms, the cardinality of the subset containing the odd-numbered rooms is the same as the cardinality of the set of all rooms. Indeed, infinite sets are characterized as sets that have proper subsets of the same cardinality. For countable sets (sets with the same cardinality as the natural numbers) this cardinality is \(\aleph_0\) (aleph-null).

Rephrased, for any countably infinite set, there exists a bijective function which maps the countably infinite set to the set of natural numbers, even if the countably infinite set contains the natural numbers. For example, the set of rational numbers—those numbers which can be written as a quotient of integers—contains the natural numbers as a subset, but is no bigger than the set of natural numbers since the rationals are countable: There is a bijection from the naturals to the rationals.

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The Grand Hotel Cigar Mystery

Another story regarding the Grand Hotel can be used to show that mathematical induction only works from an induction basis.

Suppose that the Grand Hotel does not allow smoking, and no cigars may be taken into the Hotel. Despite this, the guest in room 1 goes to the guest in room 2 to get a cigar. The guest in room 2 goes to room 3 to get two cigars—one for himself and one for the guest in room 1. In general, the guest in room N goes to room (N+1) to get N cigars. They each return, smoke one cigar and give the rest to the guest from room (N-1). Thus despite the fact no cigars have been brought into the hotel, each guest can smoke a cigar inside the property.

The fallacy of this story derives from the fact that there is no inductive point (base-case) from which the induction can derive. Although it is shown that if the guest from room N has N cigars then both he and all guests in lower-numbered rooms can smoke, it is never proved that any of the guests actually have cigars therefore it doesn't follow that any guest can smoke a cigar inside the Hotel. The fact that the story mentions that cigars are not allowed into the hotel is designed to highlight the fallacy. However, since there is an infinite number of rooms in the hotel and each guest (N) must go to guest (N+1) for his cigar, this process of going up one room never ends and no cigars are ever smoked.

References in fiction

- The novel White Light by mathematician/science fiction writer Rudy Rucker includes a hotel based on Hilbert's paradox, and where the protagonist of the story meets Georg Cantor.
- Mark Z. Danielewski's novel "House of Leaves" centers around a house containing an infinite number of rooms.
- Stephen Baxter's science fiction novel Transcendent has a brief discussion on the nature of infinity, with an explanation based on the paradox, modified to use starship troopers rather than hotels.
- Geoffrey A. Landis' Nebula Award-winning short story "Ripples in the Dirac Sea" uses the Hilbert hotel as an explanation of why an infinitely-full Dirac sea can nevertheless still accept particles.
- In Peter Høeg's novel Smilla's Sense of Snow, the titular heroine reflects that it is admirable for the hotel's manager and guests to go to all that trouble so that the latecomer can have his own room and some privacy.
- In Ivar Ekeland's novel for children, The Cat in Numberland, a "Mr. Hilbert" and his wife run an infinite hotel for all the integers. The story progresses through the triangular method for the rationals.

References


External links

- Nancy Casey, Welcome to the Hotel Infinity! (http://www.ccs3.lanl.gov/mega-math/workbk/infinity/ infinity.html) — The paradox told as a humorous narrative, featuring a hotel owner and a building contractor based on the feuding 19th-century mathematicians Georg Cantor and Leopold Kronecker
- Hilbert's Infinite Hotel (http://www.bbc.co.uk/dna/h2g2/A4080467), h2g2
- The Hilbert Hotel (http://www.youtube.com/user/Davidson1956#p/u/5/BRn_GNcglKo) - YouTube presentation
- "Beyond the Finite" (http://web.science.mq.edu.au/~chris/beyond/infinity.pdf)