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Thermoelastic stability of functionally graded cylindrical shells

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Abstract

In this paper, the problems of thermal buckling in axial direction of cylindrical shells made of functionally graded materials are discussed. Based on the Donnell's shell theory, the equilibrium and stability equations of the cylindrical shell subjected to thermal loads are derived firstly. Then the closed form solutions are presented for the shell with simply supported boundary conditions subjected to three types of thermal loading. The material properties are assumed varying as a power form of thickness coordinate variable. The influences of the aspect ratio, the relative thickness and the functionally graded index on the buckling temperature difference are carefully discussed.

Keywords: Functionally graded material; Thermal buckling; Instability of cylindrical shell

1. Introduction

The use of functionally graded materials (FGMs) has gained much popularity in recent years especially in extreme high temperature environments such as the nuclear reactor and high-speed spacecraft industries. FGMs are composite materials, which are microscopically inhomogeneous, and the mechanical properties vary smoothly or continuously from one surface to the other. This is achieved by gradually varying the volume fraction of the constituent materials. It is this continuous change in composition that results in the graded properties of FGMs. Typically, these materials are made from a mixture of metal and ceramic. The advantage of using these materials is that they are able to withstand high-temperature gradient environments while maintaining their structural integrity. The ceramic constituent of the material provides the high temperature resistance due to its low thermal conductivity. The ductile metal constituent, on the other hand, prevents fracture caused by stresses due to the high-temperature gradient in a very short period of time. Furthermore, a mixture of the ceramic and a metal with a continuously varying volume fraction can be easily manufactured [1–3]. This eliminates interface problems and thus the stress distributions are smooth.

FGMs were initially designed as thermal barrier materials for aerospace structural applications and fusion reactors. These materials are now developed for general use as structural elements in extreme high temperature environments. Many studies have examined functionally graded materials as thermal barriers. For examples, Tanaka et al. [4] presented an improved solution to thermo-elastic material design in FGMs in order to reduce thermal stresses. Obata and Noda [5] investigated the state thermal stress field in hollow circular cylinder and a hollow sphere of a functionally graded material. Tanigawa et al. [6] derived a one-dimensional temperature solution for a non-homogeneous plate in transient state and also optimized the material composition by introducing a laminated composite model. Analytical formulation and numerical solution of the thermal stresses and deformations for axisymmetrical shells of FGMs subjected to thermal loading due to fluid is obtained by Takezono et al. [7]. Zimmerman and Lutz

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analyzed the thermal stresses and deformations of FGM plates under steady graded temperature field [8]. Reddy and Chin investigated the thermal deformations of FGM plates and shells in Ref. [9]. Chen et al. [10,11] studied the stresses distribution and free vibrations of FGM plates and shells exactly using the three dimensional elastic theory via the state space method. Li and Zou [12] analyzed the stresses distribution of a FGM cylinder subjected to internal pressure load by the finite element method. Durodola and Adlington [13] presented the use of numerical methods to assess the effect of various forms of gradation of material properties to control deformations and stresses in rotating axi-symmetric components such as disks and rotors. Considerable research has also been performed on the analysis of the stresses and deformations of functionally graded structures. With the increased usage of these materials, it is also important to understand the buckling behavior of functionally graded material structures. A few studies have addressed this. A formulation of the stability problem for FGM plates was presented by Birman [14] where a micro-mechanical model was employed to solve the buckling problem for a rectangular plate subjected to uniaxial compression. Javaheri and Eslami presented the thermal buckling analysis of rectangular FGM plates based on the classical plate theory [15]. In their paper the nonlinear equilibrium and linear stability equations are derived using variational formulations, and then the closed form solutions for the linear stability equations are presented. Motivated by Javaheri, Wu examined the buckling behavior of functionally graded rectangular plates with simply supported boundary conditions under thermal loads by using the first shear deformation theory [16]. Buckling analysis for circular FGM plates under radial compressive loading and thermal loading have been presented by Najafizadeh and Eslami [17].

In all the papers mentioned above, researchers have confined their studies to FGM plates. For the thermoelastic stability analysis of cylindrical shells, Eslami and Ziaii [18] discussed the buckling behavior of thin cylindrical shells made of isotropic materials based on improved stability equations. Thangartnam [19] presented the thermal buckling formulations for laminated composite shells. Eslami [20] investigated the similar problem of composite cylindrical shells. Their studies incorporated mechanical and thermal loads simultaneously. Ganesan [21] analyzed the buckling and dynamic properties of cylindrical shells made of piezoelectric composite materials.

From the literature survey, it is seen that few studies have been made for thermal buckling analysis of cylindrical shells made of functionally graded materials. This is the motivation for this paper, the aim of which is to develop the stability equations of FGM cylinders and to discuss the buckling behavior of FGM cylindrical shells under thermal loads. Firstly, the stability equations of the shell are established by the critical equilibrium method based on the Donnell's shell theory. Then the analytical solutions of buckling equations for FGM cylinders under three types of thermal loads are presented using the Galerkin's method. The numerical results are validated against known data in the literature.

2. Material properties

The functionally graded cylindrical shell as shown in Fig. 1 is assumed to be thin and of length l, thickness h and radius R. The x-axis is taken along a generator, the circumferential arc length subtends an angle θ, and the z-axis is directed radially outwards. The modulus of elasticity E and the coefficient of thermal expansion α are assumed changing in the thickness direction z based on the Voigt's rule over the whole range of the volume fraction, whereas Poisson's ratio v is assumed to be a constant as

$$E(z) = E_c V_c + E_m V_m, \quad \alpha(z) = \alpha_c V_c + \alpha_m V_m, \quad v(z) = v \tag{1}$$

where V_m and V_c denote the volume fractions of the metal and the ceramic, respectively. They are expressed as

$$V_m = (z/h + 1/2)^k, \quad V_c = 1 - V_m \tag{2}$$

where z is the thickness coordinate; and $-h/2 \leqslant z \leqslant h/2$, where h is the thickness of the shell and k is the power law index that takes values greater than or equal to zero. Substituting Eq. (2) into Eq. (1), mechanical properties of the FGM shell are determined, which are the same as the equations proposed by Shahsiah and Eslami [22]

$$\begin{aligned} E(z) &= E_{mc}(z/h + 1/2)^k + E_c, \\ \alpha(z) &= \alpha_{mc}(z/h + 1/2)^k + \alpha_c, \\ v(z) &= v \end{aligned} \tag{3}$$

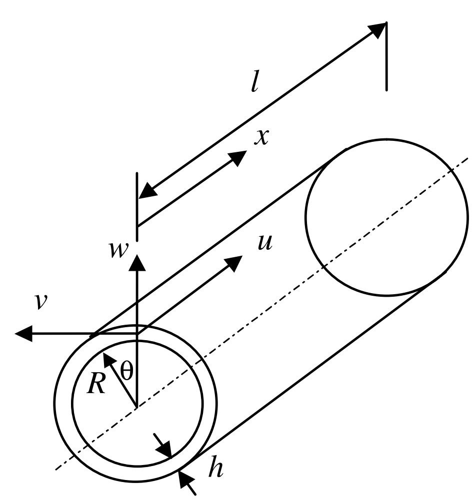

Fig. 1. Coordinate system of the FGM cylindrical shell.

where

$$E_{mc} = E_m - E_c, \quad \alpha_{mc} = \alpha_m - \alpha_c \tag{4}$$

3. Stability equations

According to Donnel's shell theory, the relations between the strains and the displacements are as

$$\varepsilon_x = u_{,x} - z w_{,xx}, \quad \varepsilon_\theta = \frac{1}{R}(v_{,\theta} + w) - \frac{z}{R^2} w_{,\theta\theta},$$
$$\gamma_{x\theta} = \frac{1}{R} u_{,\theta} + v_{,x} - \frac{2z}{R} w_{,x\theta} \tag{5}$$

where u, v, w refer to displacements in x, θ, z directions respectively. Hooke's law for a shell is defined as

$$\sigma_x = \frac{E}{1-v^2}[\varepsilon_x + v\varepsilon_\theta - (1+v)\alpha T],$$
$$\sigma_\theta = \frac{E}{1-v^2}[\varepsilon_\theta + v\varepsilon_x - (1+v)\alpha T], \quad \tau_{x\theta} = G\gamma_{x\theta} \tag{6}$$

The forces and moments per unit length of the shell expressed in terms of the stress components through the thickness are

$$N_{ij} = \int_{-h/2}^{h/2} \sigma_{ij}\,dz, \quad M_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} z\,dz \tag{7}$$

Substituting Eqs. (3), (5) and (6) into Eq. (7), gives the constitutive relations as

$$N_x = \frac{E_1}{1-v^2}\left(u_{,x} + \frac{v}{R}(v_{,\theta} + w)\right) - \frac{E_2}{1-v^2}\left(w_{,xx} + \frac{v}{R^2} w_{,\theta\theta}\right) - \frac{\Phi}{1-v}$$
$$N_\theta = \frac{E_1}{1-v^2}\left(v u_{,x} + \frac{1}{R}(v_{,\theta} + w)\right) - \frac{E_2}{1-v^2}\left(v w_{,xx} + \frac{1}{R^2} w_{,\theta\theta}\right) - \frac{\Phi}{1-v}$$
$$N_{x\theta} = \frac{E_1}{2(1+v)}\left(\frac{1}{R} u_{,\theta} + v_{,x}\right) - \frac{E_2}{(1+v)R} w_{,x\theta}$$
$$M_x = \frac{E_2}{1-v^2}\left(u_{,x} + \frac{v}{R}(v_{,\theta} + w)\right) - \frac{E_3}{1-v^2}\left(w_{,xx} + \frac{1}{R^2} v w_{,\theta\theta}\right) - \frac{\Theta}{1-v}$$
$$M_\theta = \frac{E_2}{1-v^2}\left(v u_{,x} + \frac{1}{R}(v_{,\theta} + w)\right) - \frac{E_3}{1-v^2}\left(v w_{,xx} + \frac{1}{R^2} w_{,\theta\theta}\right) - \frac{\Theta}{1-v}$$
$$M_{xy} = \frac{E_2}{2(1+v)}\left(\frac{1}{R} u_{,\theta} + v_{,x}\right) - \frac{E_3}{(1+v)R} w_{,x\theta} \tag{8}$$

where

$$E_1 = E_c h + \frac{E_{mc} h}{k+1}, \qquad E_2 = \frac{k E_{mc} h^2}{2(k+1)(k+2)},$$
$$E_3 = \frac{1}{12} E_c h^3 + E_{mc} h^3 \left[\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4(k+1)}\right]$$
$$\Phi = \int_{-h/2}^{h/2} \left[E_c + E_{mc}\left(\frac{1}{2} + \frac{z}{h}\right)^k\right] \times \left[\alpha_c + \alpha_{mc}\left(\frac{1}{2} + \frac{z}{h}\right)^k\right] T(x,y,z)\,dz$$
$$\Theta = \int_{-h/2}^{h/2} \left[E_c + E_{mc}\left(\frac{1}{2} + \frac{z}{h}\right)^k\right] \times \left[\alpha_c + \alpha_{mc}\left(\frac{1}{2} + \frac{z}{h}\right)^k\right] T(x,y,z) z\,dz \tag{9}$$

The nonlinear equations of equilibrium according to Donnell's theory are thus given by

$$N_{x,x} + \frac{1}{R} N_{x\theta,\theta} = 0, \qquad \frac{1}{R} N_{\theta,\theta} + N_{x\theta,x} = 0,$$
$$M_{x,xx} + \frac{2}{R} M_{x\theta,x\theta} + \frac{1}{R^2} M_{\theta,\theta\theta} - \frac{1}{R} N_\theta + N_x w_{,xx} + \frac{1}{R^2} N_\theta w_{,\theta\theta} + \frac{2}{R} N_{x\theta} w_{,x\theta} = 0 \tag{10}$$

When Eq. (8) are substituted into Eq. (10), the equations of equilibrium can be expressed in terms of displacement components. If the temperature difference is uniform in the x and θ directions, the equations of equilibrium are written as follows:

$$\frac{E_1}{1-v^2} u_{,xx} + \frac{E_1}{2(1+v)R^2} u_{,\theta\theta} + \frac{(1+v)E_1}{2(1-v^2)R} v_{,x\theta} + \frac{vE_1}{(1-v^2)R} w_{,x} - \frac{E_2}{1-v^2} w_{,xxx} - \frac{E_2}{(1-v^2)R^2} w_{,x\theta\theta} = 0 \tag{11}$$

$$\frac{(1+v)E_1}{2(1-v^2)R} u_{,x\theta} + \frac{E_1}{(1-v^2)R^2} v_{,\theta\theta} + \frac{E_1}{2(1+v)} v_{,xx} + \frac{E_1}{(1-v^2)R^2} w_{,\theta} - \frac{E_2}{(1-v^2)R} w_{,xx\theta} - \frac{E_2}{(1-v^2)R^3} w_{,\theta\theta\theta} = 0 \tag{12}$$

$$\frac{E_2}{1-v^2}\left(u_{,xxx} + \frac{1}{R^2} u_{,x\theta\theta}\right) + \frac{E_2}{(1-v^2)R}\left(v_{,xx\theta} + \frac{1}{R^2} v_{,\theta\theta\theta}\right) + \frac{vE_2}{(1-v^2)R} w_{,xx} + \frac{E_2}{(1-v^2)R^3} w_{,\theta\theta}$$
$$- \frac{E_3}{1-v^2}\left(w_{,xxxx} + \frac{2}{R^2} w_{,xx\theta\theta} + \frac{1}{R^4} w_{,\theta\theta\theta\theta}\right) - \frac{1}{R}\left(\frac{E_1}{(1-v^2)R} w - \frac{vE_2}{1-v^2} w_{,xx} - \frac{E_2}{(1-v^2)R^2} w_{,\theta\theta}\right)$$
$$- \frac{\Phi}{(1-v)R} + N_x w_{,xx} + \frac{2}{R} N_{x\theta} w_{,x\theta} + \frac{1}{R^2} N_\theta w_{,\theta\theta} = 0 \tag{13}$$

Substituting Eqs. (11) and (12) into Eq. (13), we can eliminate the variables u and v. Thus we have

$$\frac{E_2^2 - E_1E_3}{E_1(1-\nu^2)}\nabla^4\nabla^4 w - \frac{E_1}{R^2}w_{,xxxx} + \nabla^4\left(N_x w_{,xx} + \frac{2}{R}N_{x\theta}w_{,x\theta} + \frac{1}{R^2}N_\theta w_{,\theta\theta}\right) = 0 \tag{14}$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2}{R^2}\frac{\partial^4}{\partial x^2\partial\theta^2} + \frac{1}{R^4}\frac{\partial^4}{\partial\theta^4} \tag{15}$$

To establish the stability equations, the critical equilibrium method is used. Assuming that the state of stable equilibrium of a general circular cylindrical shell under thermal load may be designated by w_0. The displacement of the neighboring state is $w_0 + w_1$, where w_1 is an arbitrary small increment of displacement. Substituting $w_0 + w_1$ into Eq. (14) and resulting the following stability equation

$$\frac{E_2^2 - E_1E_3}{E_1(1-\nu^2)}\nabla^4\nabla^4 w_1 - \frac{E_1}{R^2}w_{1,xxxx} + \nabla^4\left(N_{x0} w_{1,xx} + \frac{2}{R}N_{x\theta 0}w_{1,x\theta} + \frac{1}{R^2}N_{\theta 0} w_{1,\theta\theta}\right) = 0 \tag{16}$$

where N_{x0}, $N_{\theta 0}$ and $N_{x\theta 0}$ are the pre-buckling thermal forces.

4. Buckling of a FGM cylindrical shell

In this section, the closed form solutions of Eq. (16) for three types of thermal loading conditions are presented. To determine the buckling temperature difference, the pre-buckling thermal forces should be found first. The shell is assumed to be simply supported in bending, free in radial expansion and rigidly supported in axial extension. The temperature varies only in the thickness direction, and remains constant in the longitudinal and circumferential directions of the shell. Thus, the prebuckling deformation should be axial-symmetric, and we have the following equations

$$u_0 = 0, \quad v_0 = 0, \quad N_{\theta 0} = N_{x\theta 0} = 0 \tag{17}$$

From these equations and the constitutive equations, one has

$$N_{x0} = -\Phi \tag{18}$$

Substituting Eqs. (17) and (18) into the stability equation (16), results in the following governing equation

$$\frac{E_2^2 - E_1E_3}{E_1(1-\nu^2)}\nabla^4\nabla^4 w_1 - \frac{E_1}{R^2}w_{1,xxxx} - \Phi\nabla^4 w_{1,xx} = 0 \tag{19}$$

Let

$$w_1 = c\sin\frac{m\pi x}{l}\sin n\theta \tag{20}$$

where c is an arbitrary coefficient; m and n are the buckling wave numbers in the axial direction and the circumferential direction, respectively. Substituting Eq. (20) into Eq. (19), results in the following equation

$$\frac{E_1E_3 - E_2^2}{E_1(1-\nu^2)}\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^4 + \frac{E_1}{R^2}\left(\frac{m\pi}{l}\right)^4 - \Phi\left(\frac{m\pi}{l}\right)^2\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2 = 0 \tag{21}$$

Then the thermal parameter Φ is expressed as

$$\Phi = \frac{E_1E_3 - E_2^2}{E_1(1-\nu^2)}\frac{\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2}{\left(\frac{m\pi}{l}\right)^2} + \frac{E_1}{R^2}\frac{\left(\frac{m\pi}{l}\right)^2}{\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2} \tag{22}$$

For minimizing the parameter Φ, we should make the first order derivative of Φ with respect to $\frac{\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2}{\left(\frac{m\pi}{l}\right)^2}$ be zero, and we have

$$\frac{\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n}{R}\right)^2\right]^2}{\left(\frac{m\pi}{l}\right)^2} = \sqrt{\frac{E_1^2(1-\nu^2)}{(E_1E_3 - E_2^2)R^2}} \tag{23}$$

Then substituting Eq. (23) into Eq. (22), the minimum value of parameter Φ is obtained

$$\Phi_{\min} = \frac{2}{R}\sqrt{\frac{E_1E_3 - E_2^2}{(1-\nu^2)}} \tag{24}$$

4.1. Uniform temperature rise

When the temperature changes uniformly through the shell thickness, the thermal parameter Φ is defined as

$$\Phi = \int_{-h/2}^{h/2}\left[E_c + E_{mc}\left(\frac{1}{2} + \frac{z}{h}\right)^k\right]\left[\alpha_c + \alpha_{mc}\left(\frac{1}{2} + \frac{z}{h}\right)^k\right] \times T(x,y,z)\,dz = P\cdot T \tag{25}$$

where

$$P = E_c\alpha_c h + \frac{E_c\alpha_{mc} + E_{mc}\alpha_c}{k+1}h + \frac{E_{mc}\alpha_{mc}}{2k+1}h \tag{26}$$

Substituting Eq. (25) into Eq. (24), yields the critical buckling temperature

$$T_{cr}^0 = \frac{2}{PR}\sqrt{\frac{E_1E_3 - E_2^2}{(1-\nu^2)}} \tag{27}$$

When the power law index is set equal to one ($k = 1$), Eq. (27) is reduced to the critical temperature difference for functionally graded shell with linear composition of ceramics and metal. Also, when the power law index is set equal to zero, Eq. (27) is reduced to the critical temperature difference of homogeneous shells

$$T_{cr}^{0} = \frac{1}{\sqrt{3(1-\nu^2)}}\frac{h}{\alpha R} \tag{28}$$

4.2. Linear temperature change

Assume that the temperature change is linear through the thickness as

$$T(z) = (\Delta T/h)(z + h/2) + T_c \tag{29}$$

where z is the coordinate variable in the thickness direction which measured from the middle plane of the shell. T_c is the ceramic temperature and ΔT is the temperature difference between metal surface and ceramic surface, i.e., $\Delta T = T_m - T_c$. For this loading case, the thermal parameter Φ can be expressed as

$$\Phi = PT_c + X \cdot \Delta T \tag{30}$$

where

$$X = E_c\alpha_c h/2 + \frac{E_c\alpha_{mc} + E_{mc}\alpha_c}{k+2}h + \frac{E_{mc}\alpha_{mc}}{2(k+1)}h \tag{31}$$

From Eq. (30) one has

$$\Delta T = \frac{\Phi - PT_c}{X} = \frac{\Phi - PT_m}{X - P} \tag{32}$$

Substituting Eq. (24) into Eq. (32), we obtain the critical temperature difference

$$T_{cr}^{1} = -\Delta T_{cr} = \frac{2}{(P-X)R}\sqrt{\frac{E_1E_3 - E_2^2}{(1-\nu^2)}} - \frac{PT_m}{P-X} \tag{33}$$

4.3. Nonlinear temperature change

The functionally graded materials are designed in order to resist high temperature rise by ceramic, so the temperature change will be quite different at the two sides of the FGM structures. When the temperature rises differently at the inner and outer surfaces of the shell, the temperature distribution across the thickness is governed by the steady state heat conduction equation and boundary condition as follows:

$$\frac{d}{dz}\left[K(z)\frac{dT}{dz}\right] = 0, \quad T = T_m \ (\text{at } z = h/2),$$
$$T = T_c \ (\text{at } z = -h/2) \tag{34}$$

where $K(z)$ is the coefficient of thermal conduction. Similar to the elasticity and thermal expansion properties, we assume that the thermal conductive coefficient is also a power form function as

$$K(z) = K_{mc}(z/h + 1/2)^k + K_c \tag{35}$$

where

$$K_{mc} = K_m - K_c \tag{36}$$

The solution of Eq. (34) is obtained by means of polynomial series. Taking the first seven terms of the series, the solution for temperature distribution across the shell thickness becomes

$$\begin{aligned} T(z) = T_c &+ \frac{\Delta T}{C}\left[\left(\frac{2z+h}{2h}\right) - \frac{K_{mc}}{(k+1)K_c}\left(\frac{2z+h}{2h}\right)^{k+1} + \frac{K_{mc}^2}{(2k+1)K_c^2}\left(\frac{2z+h}{2h}\right)^{2k+1}\right] \\ &+ \frac{\Delta T}{C}\left[-\frac{K_{mc}^3}{(3k+1)K_c^3}\left(\frac{2z+h}{2h}\right)^{3k+1} + \frac{K_{mc}^4}{(4k+1)K_c^4}\left(\frac{2z+h}{2h}\right)^{4k+1} - \frac{K_{mc}^5}{(5k+1)K_c^5}\left(\frac{2z+h}{2h}\right)^{5k+1}\right] \end{aligned} \tag{37}$$

with

$$C = 1 - \frac{K_{mc}}{(k+1)K_c} + \frac{K_{mc}^2}{(2k+1)K_c^2} - \frac{K_{mc}^3}{(3k+1)K_c^3} + \frac{K_{mc}^4}{(4k+1)K_c^4} - \frac{K_{mc}^5}{(5k+1)K_c^5} \tag{38}$$

where $\Delta T = T_m - T_c$ is defined as the temperature difference between ceramic-rich and metal-rich surfaces of the shell. Substituting Eq. (37) into the thermal parameter equation (9), yields

$$\Phi = PT_c + H \cdot \Delta T \tag{39}$$

where

$$\begin{aligned} H = \frac{1}{C}\{&E_c\alpha_c[h/2 - K_{mc}h/(k+1)/(k+2)/K_c + K_{mc}^2h/(2k+1)/(2k+2)/K_c^2 - K_{mc}^3h/(3k+1)/(3k+2)/K_c^3 \\ &+ K_{mc}^4h/(4k+1)/(4k+2)/K_c^4 - K_{mc}^5h/(5k+1)/(5k+2)/K_c^5] + (E_c\alpha_{mc} + E_{mc}\alpha_c)[h/(k+2) \\ &- K_{mc}h/(k+1)/(2k+2)/K_c + K_{mc}^2h/(2k+1)/(3k+2)/K_c^2 - K_{mc}^3h/(3k+1)/(4k+2)/K_c^3 \\ &+ K_{mc}^4h/(4k+1)/(5k+2)/K_c^4 - K_{mc}^5h/(5k+1)/(6k+2)/K_c^5] + E_{mc}\alpha_{mc}[h/(2k+2) \\ &- K_{mc}h/(k+1)/(3k+2)/K_c + K_{mc}^2h/(2k+1)/(4k+2)/K_c^2 - K_{mc}^3h/(3k+1)/(5k+2)/K_c^3 \\ &+ K_{mc}^4h/(4k+1)/(6k+2)/K_c^4 - K_{mc}^5h/(5k+1)/(7k+2)/K_c^5]\} \end{aligned} \tag{40}$$

From Eq. (39) one has

$$\Delta T=\frac{\Phi-PT_{c}}{H}=\frac{\Phi-PT_{m}}{H-P} \tag{41}$$

Substituting Eq. (24) into Eq. (41), we obtain the critical temperature difference

$$T_{cr}^{2}=-\Delta T_{cr}=\frac{2}{R(P-H)}\sqrt{\frac{E_1E_3-E_2^2}{(1-\nu^2)}}-\frac{PT_{m}}{P-H} \tag{42}$$

It should be noted that Eqs. (27), (33) and (42) can only be used to determine the buckling parameters Φ for cylinders which having long geometric shapes, because Eq. (23) is not suitable for short shells. From Eq. (23) we can obtain

$$\beta=\frac{nl}{\pi R}=\left\{\left[\left(\frac{E_1^2(1-\nu^2)l^4}{\pi^4R^2(E_1E_3-E_2^2)}\right)\right]^{\frac{1}{4}}m-m^2\right\}^{\frac{1}{2}} \tag{43}$$

Since m is greater than 1 and β cannot be imagination, the parameter in bracket must be greater than 1. Thus we have

$$Z=\frac{E_1^2(1-\nu^2)l^4}{\pi^4R^2(E_1E_3-E_2^2)}\geqslant 1 \tag{44}$$

So we can conclude that Eqs. (27), (33) and (42) are only suitable for long shells that satisfying Eq. (44). For the shells with short length that makes the parameter Z less than 1, the minimum value of thermal parameter Φ should be determined by taking $m = 1$, $n = 0$ in Eq. (22). In these cases, we have

$$\Phi_{\min}=\frac{\pi^2(E_1E_3-E_2^2)}{E_1l^2(1-\nu^2)}+\frac{E_1l^2}{\pi^2R^2} \tag{45}$$

For these short shells, the critical buckling temperature difference should be determined by the following three equations with uniform, linear and nonlinear temperature loading cases

$$\begin{aligned}
T_{cr}^{0}&=\frac{\pi^2(E_1E_3-E_2^2)}{E_1Pl^2(1-\nu^2)}+\frac{E_1l^2}{\pi^2R^2P}\\
T_{cr}^{1}&=\frac{1}{P-X}\left[\frac{\pi^2(E_1E_3-E_2^2)}{E_1l^2(1-\nu^2)}+\frac{E_1l^2}{\pi^2R^2}-PT_{m}\right]\\
T_{cr}^{2}&=\frac{1}{P-H}\left[\frac{\pi^2(E_1E_3-E_2^2)}{E_1^2(1-\nu^2)}+\frac{E_1l^2}{\pi^2R^2}-PT_{m}\right]
\end{aligned} \tag{46}$$

5. Numerical results and discussion

To illustrate the proposed method, a ceramic-metal functionally graded cylindrical shell is considered. The combination of materials consists of aluminum and alumina. The coefficients of Yong's modulus, conductivity, and thermal expansion for alumina are $E_c = 380$ GPa, $K_c = 10.4$ W/mK, $\alpha_c = 7.4\times10^{-6}$ (1/°C), and for aluminum are $E_m = 70$ GPa, $K_m = 204$ W/mK, $\alpha_m = 23\times10^{-6}$ (1/°C), respectively. Poisson's ratio is chosen as $\nu = 0.3$.

Firstly, The critical buckling temperature or temperature differences T_{cr} with respect to the relative thickness h/R are calculated for functionally graded shells with different volume fraction exponent under uniform temperature rise, linear and nonlinear temperature distribution across the thickness and are plotted in Figs. 2 and 3. These two figures show that the critical buckling temperature or temperature difference T_{cr} increases linearly as the relative thickness h/R increases, whatever the gradient index k is. It is seen from Fig. 2 that the values of critical buckling temperature difference for homogeneous shells ($k = 0$) calculated with linear or nonlinear temperature distribution assumption are identical as expected. Actually, as the temperature changes differently at inner and outer surface of a homogeneous cylindrical shell, the temperature distribution must be linear across the thickness for thin shells. From Fig. 3 it is found that

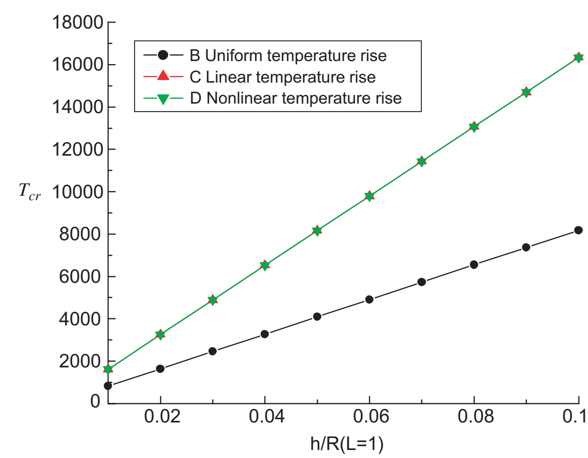

Fig. 2. Critical buckling temperature rise of a functionally graded cylindrical shell vs h/R ($k = 0$).

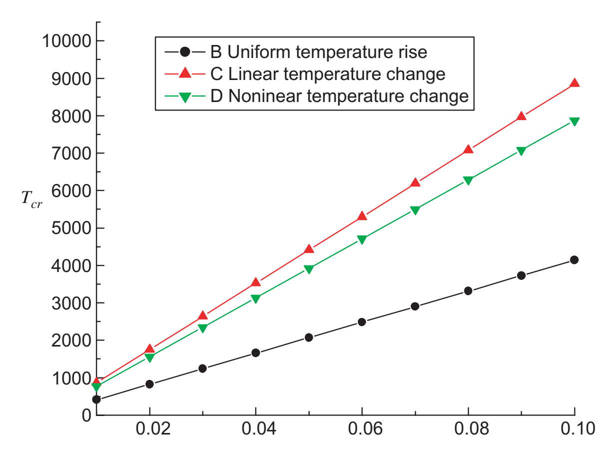

Fig. 3. Critical buckling temperature rise of a functionally graded cylindrical shell vs h/R ($k = 1$).

for a functionally graded cylindrical shell with $k = 1$, the values of the buckling temperature difference computed with linear temperature distribution assumption across the thickness are higher than those computed by nonlinear temperature distribution assumption. It is also found that the thicker the shell, the larger the difference between the buckling temperature of linear and nonlinear temperature distributions will be.

Figs. 4 and 5 demonstrate the variation trends of critical buckling temperature for cylinders with different material gradient indexes vs the aspect ratio R/l of the shell. It is obvious that as R/l increases from 1 to 8, the critical buckling temperature reduces rapidly whether for $k = 0.5$ or for $k = 1$. However, when R/l is greater than 8, the critical buckling temperature changes very slowly. This means that as the shell is short enough, the critical buckling temperature will remain a constant.

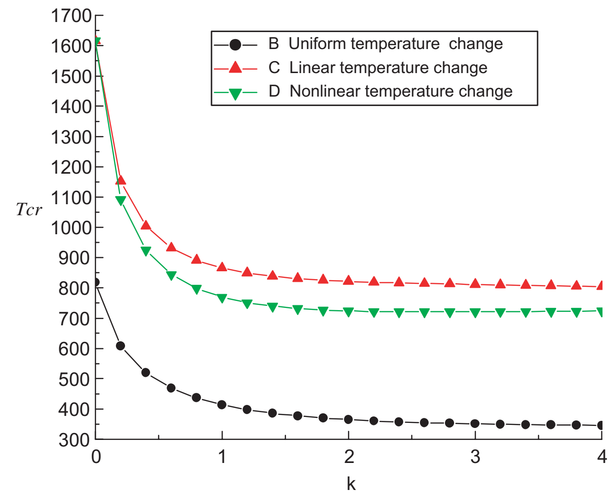

Fig. 6. Critical buckling temperature rise of a functionally graded cylindrical shell vs k ($l = R = 1$, $h = 0.01$).

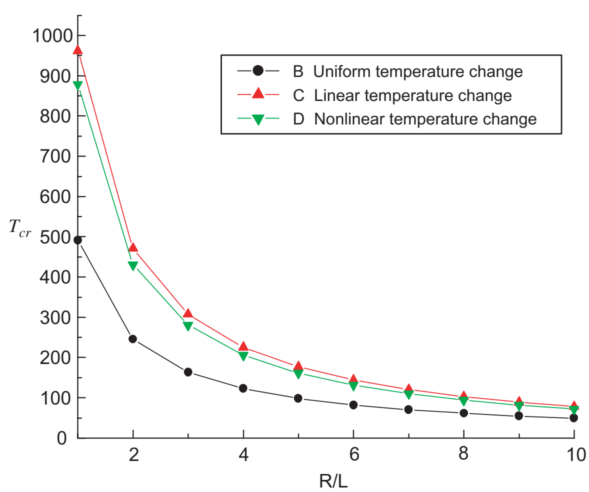

Fig. 4. Critical buckling temperature rise of a functionally graded cylindrical shell vs R/l ($k = 0.5$).

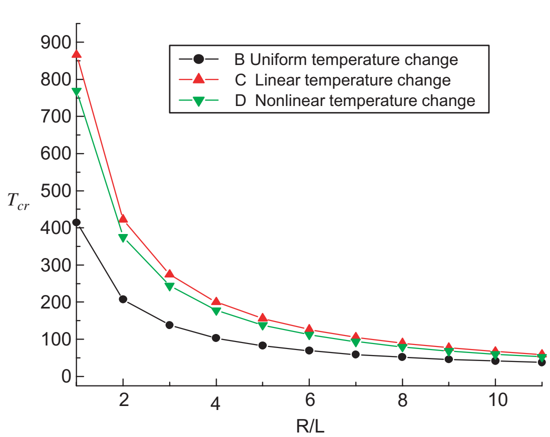

Fig. 5. Critical buckling temperature rise of a functionally graded cylindrical shell vs R/l ($k = 1.0$).

Fig. 6 shows the buckling temperature vs the material gradient exponent k for a shell with $h = 0.01$, $l = R = 1$. We can see that the critical buckling temperature for a homogeneous ceramic cylinder with $k = 0$ is considerably higher than those for the functionally graded cylinders with $k \geqslant 0$. It is evident that the buckling temperature decreases as the material volume fraction exponent k increases monotonically. As the gradient index k changes from 0 to 1, the critical buckling temperature decreases significantly. When k changes from 1 to 2, it reduces very slowly, and as k becomes larger than 2, it will be a constant practically. It is also found that for a FGM cylinder with small gradient index k, the difference between the buckling temperature of linear and nonlinear temperature distributions is very small.

In order to ascertain the present method, comparison studies are carried out for the buckling temperature difference with those obtained by Shahsiah and Eslami [22] using the improved Donnell shell theory. A long shell with $l/R = 10$ is considered. The shell is made of steel and alumina with material properties $E_c = 380\,\text{GPa}$, $K_c = 10.4\,\text{W/mK}$, $\alpha_c = 7.4 \times 10^{-6}$ (1/°C), $E_m = 200\,\text{GPa}$, $K_m = 204\,\text{W/mK}$, $\alpha_m = 11.7 \times 10^{-6}$ (1/°C), $\nu = 0.3$. The inner surface of the shell is made of full metal. It should be noted that this shell is not in accordance with the present model, of which the inner surface is full ceramic. So, for convenience of comparison, we assume $E_m = 380\,\text{GPa}$, $K_m = 10.4\,\text{W/mK}$, $E_c = 200\,\text{GPa}$, $K_c = 204\,\text{W/mK}$, $\alpha_m = 7.4 \times 10^{-6}$ (1/°C), $\alpha_c = 11.7 \times 10^{-6}$ (1/°C), $\nu = 0.3$ in the computation. The gradient index k is taken as 1. Figs. 7 and 8 plot the critical temperature curves for uniform temperature rise and linear temperature change, respectively. It is evident that the present results are in good agreement with those of Shahsiah, for both the case of uniform temperature rise and the linear temperature change. It is also found that the present results are

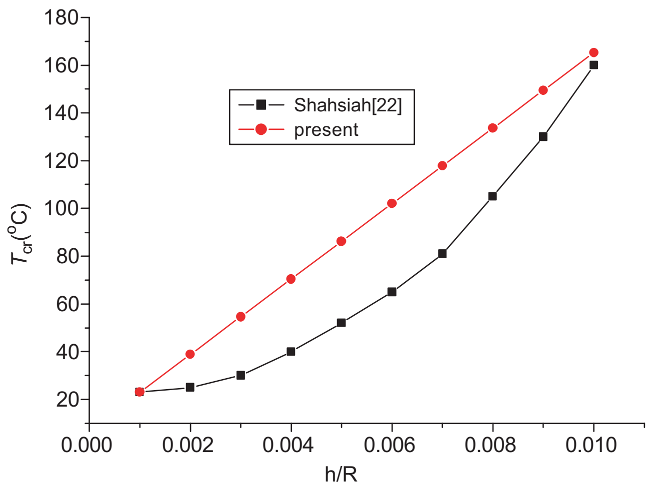

Fig. 7. Comparisons of the critical temperature under uniform temperature rise.

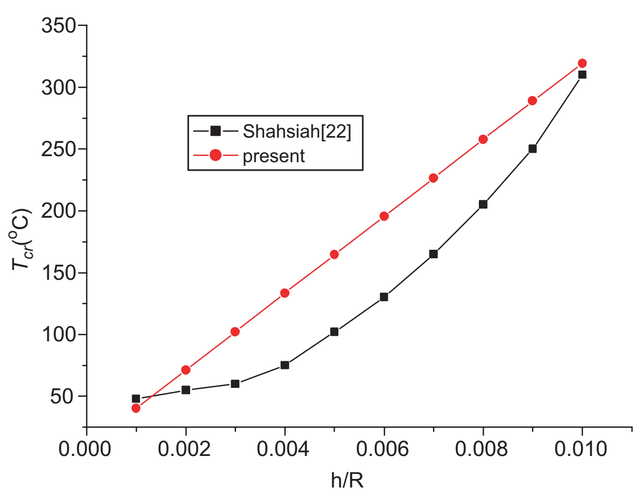

Fig. 8. Comparisons of the critical temperature under linear temperature change.

linear with respect to the relative thickness ratio h/R. However, Shahsiah's results are not linear. From Eqs. (27), (33), (42) and (46), we know that the critical temperature change is linear with respect to h/R for long shells; however, it is not linear for short shells. One can easily find that all the present results are slightly higher than those available in Ref. [22] for very thin cylindrical shells.

6. Conclusions

Circular cylindrical shells are widely used in structural design problems. When such a member is subjected to a thermal environment, its thermal buckling capacity is important in the design stage. For a design engineer, the closed form solutions for the buckling temperature of such a member is essential because the design may be quickly checked. In the present paper, Equilibrium and stability equations for a simply supported thin cylindrical shell made of functionally graded materials under thermal loads are obtained using the classical shell theory, with the assumption of power law composition for the constituent materials. Then the buckling analysis of functionally graded cylindrical shells under three types of thermal loadings is presented. Closed form solutions for the critical buckling temperature differences of shells are presented. Based on the numerical results, the following conclusions are reached:

(1) The critical buckling temperature T_{cr} for functionally graded cylindrical shells are generally lower than the corresponding values for homogeneous shells. It is very important to check the strength of the functionally graded plate due to thermal buckling, although it has many advantages as a heat resistant material.
(2) The critical buckling temperature difference T_{cr} for a functionally graded shell is increased linearly when the thickness to radius ratio increases.
(3) The critical buckling temperature difference T_{cr} for a functionally graded shell decreased by increasing the power law index k.
(4) The critical buckling temperature difference T_{cr} for a functionally graded shell decreased by increasing the radius to span ratio R/l.

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